# Announcements

1) HW #4 due next Thursday

## Example 1 :

$f(x,y) = x^2y^4$,

find absolute maximum and minimum on the unit disk :

$\{(x,y) \mid x^2+y^2 \leq 1\}$

Boundary : unit circle

$x^2+y^2=1$.

1) Find partials.

$\frac{\partial f}{\partial x} = 2xy^4$, $\frac{\partial f}{\partial y} = 4x^2y^3$

$(0,y)$ or $(x,0)$

all give

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} = 0.$$

2) Boundary: $x^2+y^2=1$, so

$$x^2 = 1-y^2.$$

Plug into $f(x,y) = x^2y^4$

to give

$$g(y) = (1-y^2)y^4$$
$$= y^4 - y^6$$

Apply Calc I max/min to g':

$$g'(y) = 4y^3 - 6y^5,$$

set equal to zero.

$$0 = 4y^3 - 6y^5$$
$$= 2y^3(2 - 3y^2)$$

$\boxed{y=0}$ or $2 - 3y^2 = 0$

$$y = \underbrace{\pm\sqrt{\frac{2}{3}}}_{\text{new}}$$

already found

Check endpoints:

since $x^2+y^2=1$,

$-1 \le y \le 1$, so

$y=-1$, $y=1$ are the

endpoints.

3) Plug all points back in!

For 1), we found $(0, y)$ and $(x, 0)$.

Plug these into $g(x,y) = x^2 y^4$, we always get zero.

From 2), we found the points $y = \pm 1$, $y = \pm\sqrt{\frac{2}{3}}$

Plug into

$$g(y) = y^4 - y^6$$

$$g(1) = g(-1) = 0$$

$$g\left(\sqrt{\frac{2}{3}}\right) = g\left(-\sqrt{\frac{2}{3}}\right)$$

$$= \frac{4}{9} - \frac{8}{27}$$

$$= \frac{4}{27} \neq 0$$

Smallest number we found

is $0$ ← absolute min

Largest number we found

is $\frac{4}{27}$ ← absolute max

# Differentiability

Consider $z = f(x,y)$.

We say $f$ is differentiable at a point $(a,b)$ in $\mathbb{R}^2$ if

$$\boxed{\lim_{(x,y)\to(a,b)} \frac{f(x,y)-f(a,b)}{\|(x,y)-(a,b)\|}}$$

exists

If $f$ is differentiable at $(a,b)$, then by holding $x=a$ or $y=b$, respectively, and taking the limit of the other variable, we get that

$$\frac{\partial f}{\partial x}(a,b), \ \frac{\partial f}{\partial y}(a,b) \text{ exist.}$$

# Connection to Partials

If $\frac{\partial f}{\partial x}(a,b)$ and $\frac{\partial f}{\partial y}(a,b)$

both exist and are continuous at $(a,b)$, then

$f$ is differentiable at

$(a,b)$.

# Absolute Maxima and Minima

Recall: Extreme Value Theorem:

any continuous function on a closed interval attains its maximum and minimum.

We generalized this to functions $z = f(x,y)$ on closed and bounded regions in $\mathbb{R}^2$.

# Closed and Bounded Regions in $\mathbb{R}^n$

A subset $S$ of $\mathbb{R}^n$ is said to be **bounded** if for all $x = (x_1, x_2, \cdots, x_n)$ in $S$, there is a number $r \geq 0$ such that

$$\boxed{\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \leq r}$$

In $\mathbb{R}^3$: $S$ bounded means there is a sphere about the origin of some finite radius that contains $S$.

A subset $S$ of $\mathbb{R}^n$ is said to be closed if it contains its "boundary".

Examples: solid spheres in $\mathbb{R}^3$

Not closed: $\{(x,y,z) \mid x > 0\}$.

# Extreme Value Theorem in $\mathbb{R}^n$

Given a real-valued function $f: \mathbb{R}^n \to \mathbb{R}$, if $f$ is continuous, then $f$ attains its maximum and minimum on any closed and bounded region $D$ in $\mathbb{R}^n$.

Example 2: (#43, section 14.7)

Find three positive numbers whose sum is 100 and whose product is a maximum.

Let's call the positive numbers $a, b$, and $c$.

product: $abc = f(a,b,c)$

sum: $a+b+c = 100$.

1) Find $\nabla f$.

$$f(a,b,c) = abc$$

$$\frac{\partial f}{\partial a} = bc, \quad \frac{\partial f}{\partial b} = ca, \quad \frac{\partial f}{\partial c} = ab$$

Set all partials equal to zero:

$$0 = bc = ca = ab$$

Can't happen if $a, b, c$ are positive.

If we permitted a, b, or c to be zero, we'd get solutions, but then the product $abc = 0$, minimum, not a maximum.

2) Boundary condition

$$a + b + c = 100$$

Solving for a,

$$a = 100 - b - c$$

$$f(a,b,c) = abc$$

$$= (100-b-c)(bc)$$

$$= 100bc - b^2c - c^2b$$

$$= g(b,c)$$

Take gradient of g

$$\frac{\partial g}{\partial b} = 100c - 2bc - c^2$$

$$\frac{\partial g}{\partial c} = 100b - b^2 - 2bc$$

Set partials equal to zero

$$0 = 100c - 2bc - c^2$$
$$= 100b - 2bc - b^2$$

$$0 = 100c - 2bc - c^2$$

since $c \neq 0$, divide by $c$ to get

$$0 = 100 - 2b - c$$

$$c = 100 - 2b$$

Plug into $0 = 100b - 2bc - b^2$

$$0 = 100b - 2b(100-2b) - b^2$$

$$0 = 100 - 2(100-2b) - b$$

$$= -100 + 3b$$

$$3b = 100, \quad b = \frac{100}{3}$$

$$c = 100 - 2b$$

$$= 100 - \frac{200}{3} = \frac{100}{3}$$

$$a + b + c = 100, \text{ so } a = \frac{100}{3}$$

Already covered the remaining boundary points in 1), so

3) $\boxed{a = b = c = \frac{100}{3}}$

# General Procedure

For finding the absolute max and min for a continuous function $f$ on a closed and bounded set $D$,

1) Find all critical points of $f$ in $D$

2) Take the equation for the boundary of $D$, plug into $f$. This reduces the number of variables of $f$ by at least one and obtains a function $g$. Find all critical points of $g$.

3) Take all points found in 1) and 2), plug back into f. Largest value = absolute max, smallest value = absolute min.